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A General Scheme for Microscopic Theories

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Abstract

We discuss from an operational point of view some fundamental concepts of the microscopic physical theories, with the aim of providing a background for a successive investigation of the microscopic space-time structure. We consistently develop the remark that a frame of reference is determined by physical objects which may interact with the objects under investigation. As it is not clear that a state can be prepared by means of physical operations, we do not use the concept of physical state for the foundation of the theory. In our approach, the primitive concepts are the measurement procedures, following which one gets a numerical result, and the transformation procedures, which have the aim of building a frame of reference. We discuss several rules which allow us to define new procedures in terms of known procedures. The statistical laws of physics are formulated in terms of an order relation between measurement procedures, which defines also an equivalence relation. The equivalence classes of measurement procedures are called measurements. We define also equivalence classes of transformation procedures, called transformations. The mathematical structure of the set of measurements and of the set of transformations is discussed in detail. We consider measurements with an arbitrary finite number of possible results, as this enables us to give a rigorous definition of compatibility. Finally, we point out that all the physical theories necessarily contain ideal measurements and transformations which do not correspond to any known physical procedure. The introduction of these ideal objects permits a considerable simplification of the mathematical structure of the theory, but reduces its physical content.

1. *Introduction*

The present paper contains a discussion of a general scheme of physical theory, which has been developed as a preliminary to an investigation of the space-time structure, at the microscopic level of elementary particles. This preliminary study is necessary because the introduction of the fundamental concepts of a physical theory necessarily implies explicit or implicit assumptions on the structure of space-time. The systematic discussion of the space-time structure will be given elsewhere. Here we only consider this argument in order to clarify our motivations.

Our point of view will be operational (Bridgman, 1927), i.e. we shall try to define our primitive concepts in terms of physical operations. This point of view permits a clear approach to many fields of natural sciences, it being almost unavoidable in microscopic physics as the microscopic objects are observed in a very indirect way, through complicated operations, which often imply refined technologies. In order to obtain a theory with a clear physical interpretation one has to introduce a set of primitive concepts directly related to the physical operations, while the other concepts (e.g. the concept of a microscopic object or of a space-time region) should be defined in terms of primitive concepts. These ideas are clearly explained in Giles (1970).

It should be clear that we are not suggesting to eliminate from physical theories all concepts which have not a direct operational meaning. As we shall see, they are unavoidable and very useful. However, they should not be confused with the primitive concepts mentioned above and their relations with the primitive concepts should be clearly understood.

As the primitive concepts of a theory have to be universal, they will not denote physical operations actually performed at a certain time in a certain place, but rather sets of prescriptions, following which the experimenter can perform certain operations. These sets of prescriptions will be called 'procedures'. They are the primitive concepts of our theory and will be discussed in detail in Section 2. An 'experiment' is a set of physical operations, actually performed following the instructions of a procedure, in certain given space-time conditions specified by means of a frame of reference.

Since the advent of relativity, it has been clear that a frame of reference has to be built by means of physical objects, e.g. rigid bodies, clocks or light rays. However, these physical objects are very often assumed to have strongly idealised properties. For instance, they are not allowed to interact directly with the physical system under investigation and their interaction with the measuring instruments is assumed to have a classical (non-quantum) nature. It is clear that the space-time structure of a theory of this kind is borrowed from some macroscopic theory, as classical physics, special relativity or general relativity.

These assumptions on the physical nature of the frames of reference can hardly be justified when we deal with very small distances and time intervals and we shall avoid them in our treatment. On the contrary, we shall consider the physical objects which define a frame of reference on the same footing as the physical system under investigation, namely we shall take into account their quantum nature, the physical limitations to their mechanical properties and so on.

We shall use the term 'situation' in order to indicate those material objects which determine the space-time conditions of an experiment. In general, a situation cannot determine a frame of reference with absolute precision. For instance, as clearly discussed in Ferretti (1968), the position and the velocity of a frame of reference can be determined exactly only by an object with infinite mass. It is conceivable that the physical properties of material objects impose some impassable limits to this precision and that the concepts of frame of reference and of inhomogeneous Lorentz transformation have to be modified in a substantial way. Of course, one should formulate a 'correspondence principle' connecting this generalised space-time structure with the known macroscopic geometry. All these arguments will be discussed elsewhere. Here we have to be careful in constructing a general scheme which does not prejudge the solution of these problems.

A short discussion is also required on the concept of a physical system, which plays a fundamental role in the elementary quantum theory, as it permits the separation between the observer and the observed object. This concept becomes rather unclear in the domain of high energy elementary particles, because, starting with a given system of particles, one can produce, by means of physical operations involving a sufficient amount of energy, another arbitrary system of particles (together with the corresponding anti-particles). It is then convenient to replace, in the above-mentioned role, the concept of a physical system by the concept of a space-time region.

From a macroscopic point of view (i.e. disregarding microscopic details) a situation defines a space-time region containing the points which can be reached by a signal leaving the objects forming the situation after the origin of the time scale. The experiments performed in the given situation can explore only this region of space-time. We see that a situation, besides defining a frame of reference, specifies also the part of the physical world which plays the role of the observed system. Therefore, an independently defined concept of a physical system is not necessary any more.

Another class of dangerous idealisations concerns the physical objects which are used for the transmission and the storage of information. For instance, it is dangerous to assume that a measurement can be performed in a bounded space-time region, as the physical objects which transmit the result could undergo an interaction outside this region. This point is very delicate if the space-time region is microscopic, as in this case the objects which transmit the information are necessarily microscopic, at least initially. We have not analysed these difficulties in detail, but we shall indicate the points where they could be relevant.

After the formalisation given by von Neumann (1932) of the general scheme of quantum mechanics, many generalisations and modifications have been proposed. One of the aims of these investigations was to formulate the basic assumptions separating, as clearly as possible, the assumptions which have a direct and natural physical justification from the assumptions which are peculiar of quantum mechanics and are justified only indirectly by its success. Considering only the first class of assumptions, we get a class of formalism which we call 'general quantum theories' (Giles, 1970; Birkhoff & von Neumann, 1936; Segal, 1947; Mackey, 1963; Haag & Kastler, 1964; Jauch, 1968; Varadarajan, 1968; Ludwig, 1970). The formalism described in the present paper is of this kind.

In the historical evolution of quantum theories one can observe a shift of attention from very idealised kinds of procedures to more and more realistic ones. It is our purpose to proceed in this direction. For instance, the measurement procedures, i.e. the procedures which have the purpose of giving a numerical result, were first described by means of the idealised concept of observable corresponding to a self-adjoint operator (von Neumann, 1932). Much later it was pointed out that not all the self-adjoint operators represent observables, due to the existence of superselection rules (Wick, Wightman & Wigner, 1952). In the framework of relativistic quantum theory, a further step has been the remark that every experiment has to be performed in a bounded spacetime region and, therefore, only operators belonging to certain local algebras can represent physical observables (Haag & Kastler, 1964; Haag, 1958). These specifications and restrictions of the set of physical observables give rise to a more accurate description of the microscopic phenomena and increase the physical content of the theory.

It has also been remarked that the observables with an infinite spectrum represent an idealisation, because the actual measurement procedures always have a finite set of possible results. Most of the modern formulations of general quantum theory consider only observables with two possible results, called propositions (Birkhoff & von Neumann, 1936; Jauch, 1968), questions (Mackey, 1963), decision effects (Ludwig, 1970) or pure tests (Giles, 1970). In the usual quantum theory (von Neumann, 1932) they are represented by means of projection operators.

The next step is due to Ludwig (1970). Performing an analysis of the real measurement procedures, he has shown that the decision effects are a very special case of a more general kind of effects which can be represented by means of positive operators with norm not larger than one. Many measurement procedures with two possible results correspond to effects of the general kind.

A deep analysis of these concepts has been given by Giles (1970), starting from an operational definition of the weighted mean of two effects (he calls them tests). The weighted mean of two tests gives rise to a mixed test in the same way as the weighted mean of two states gives rise to a mixed state. A mixed test describes a measurement procedure which contains some random choices. As these random choices can be used in order to simulate the experimental errors, the mixed tests can represent realistic measurement procedures, while the pure tests (namely the decision effects) have to be considered as idealised concepts.† We remember that the need for a mathematical formalism, taking deeply into account the experimental limitations of the accuracy of measurements, was stressed by Bridgman (1927).

The specification of the mixed tests which correspond to realistic measurement procedures leads to considerable enrichment of the theory, in particular it permits clear treatment of the topological properties of a physical system (Giles, 1970) (for instance, the number of degrees of freedom). This is by no

† This problem is also shortly discussed in note 126 of von Neumann (1932).

means unexpected: the deep connection between the concepts of general topology and the approximate nature of the experimental determinations has been discussed by Poincaré (1902 1908). Similar ideas can be found in Bourbaki (1965). These considerations hold for both quantum and classical physics.

The concept of a mixed test has a fundamental role in the formalism we shall describe. In general, we shall define in Section 2 the weighted mean of procedures of an arbitrary kind. We shall consider measurement procedures with an arbitrary finite number of possible results, for in this way we shall be able to give, in Section 6, a completely general and satisfactory treatment of the compatibility of measurements.

Besides the measurement procedures, whose only aim is to find a numerical result, we consider also 'transformation procedures', with the aim of constructing a situation. The transformation procedures transform the pre-existing situations into new situations and therefore they provide an operationally well-defined generalization of the inhomogeneous Lorentz transformations which act on the idealised frames of reference. This generalisation seems to be a promising starting point for a deep analysis of the concepts connected with the structure of space-time.

Another basic concept in almost all the approaches to quantum theory is the concept of a physical state. A physical state is defined in terms of the procedure used to prepare it. A preparation procedure is assumed to have very peculiar properties: the probabilities of the results of any measurement procedure performed after the preparation procedure are univocally determined and do not depend, for instance, on what happened before the beginning of the preparation procedure. It is well known (Haag & Kastler, 1964) that preparation procedures with this property cannot exist in relativistic quantum field theory. In general, the existence of preparation procedures is rather doubtful in any theory which deals with infinitely extended systems, as by performing a procedure one can control only a finite region of space. Even if we deal with a spatially bounded system, the condition which characterises a preparation procedure can hardly be satisfied exactly.

After these remarks, it is clear that the concepts of a physical state or of a preparation procedure cannot be used as primitive concepts in our theory. In the usual approaches to quantum theory, the concept of a preparation procedure is used in an essential way in order to formulate the empirical statistical laws of quantum physics. In fact these laws fix the probability (univocally defined) of obtaining a given result if we perform a given measurement procedure after a given preparation procedure. It is also usual to call 'equivalent' two measurement procedures if, when we perform them after the same arbitrary preparation procedure, we get the same results with the same probabilities.

In our formalism we avoid completely the use of preparation procedures and of physical states and we formulate the empirical statistical laws of the theory directly by means of an order relation defined in the set of the measurement procedures with two possible results. In Section 3 we give a direct operational definition of this order relation and we discuss its properties in detail. Starting from the order relation, we define also an equivalence relation between measurement procedures with an arbitrary number of possible results.

In order to test experimentally the order relation between two measurement procedures, we have to perform these procedures a large number of times in different situations. Therefore, we have to assume that one can build an arbitrary number of situations in such a way that experiments performed in these different situations do not disturb each other. We say that these situations are sufficiently separated in space-time. This is an unavoidable preliminary assumption about space-time. We remark that it has a large-distance asymptotic character and it does not prejudge the microscopic structure of space-time.

In Sections 4–7 we give a mathematical elaboration of the concepts operationally defined in the preceding sections. In particular, we shall consider equivalence classes of measurement procedures and of transformation procedures, which we call, respectively, 'measurements' and 'transformations'. In the current physical theories, these equivalence classes, and not the procedures themselves, are represented by means of mathematical objects (for instance by operators).

We define some algebraic operations involving measurements and transformations and, in this way, we obtain an algebraic structure. This mathematical structure is, in general, considerably more complicated than the one which appears in the usual theories. We get essential simplifications if we extend this structure by introducing some new 'ideal' measurements and transformations which do not correspond to any known procedure. In Section 7 we remark that ideal objects of this kind are present in all physical theories and that they play an essential role in the development of physics.

It is important to note that in the present paper we analyse only part of the basic concepts of a microscopic physical theory and, as a consequence, the general scheme that we propose is still too poor for a reasonably complete formulation of elementary particle physics. For instance, in order to treat a scattering process, one should consider measurement procedures composed of simpler measurement procedures performed in distant regions of space-time. We think that it is expedient to analyse first the concepts treated in the present paper.

2. *Measurement Procedures and Transformation Procedures*

By 'procedure' we mean a set of well-defined prescriptions, described in a document,† according to which the experimenter performs some physical or

† It is evident that this document is meaningful only within a given linguistic and technological context. It follows that a deeper discussion of this matter could not be confined to the domain of physical sciences.

mathematical operations which permit him to attain certain well-defined aims. We shall distinguish different 'kinds' of procedures according to the aims to be attained. In order to reduce, as far as possible, the difficulties connected with the complex and insufficiently known structure of the experimenter, the most delicate operations should be performed by means of automatic instruments. The document mentioned above should give instructions for constructing these instruments.

The following remarks are essential.

(a) If, following the prescriptions of two different documents, one necessarily performs the same physical and mathematical operations, the two documents define the same procedure. We shall also consider as equal two procedures which differ in some details when these details are known to be irrelevant on the basis of well-established theories.
(b) The prescriptions must be formulated in such a way that their aims are always attained. In particular, they should never imply infinite sets of operations.
(c) The prescriptions defining a procedure necessarily refer to some pre-existent material objects. Some of these objects define a local frame of reference and others form a device producing an event which defines the time zero.† We say that these material objects form a 'situation'. The operational prescriptions for the construction of a situation will be discussed, in a preliminary way, at the end of this section. A set of operations actually performed following a given procedure and using a given situation will be called an 'experiment'.

    A procedure in general requires some preparatory operations to be performed much in advance with respect to the time zero of the situation, for instance in order to prepare some instruments or to identify the objects which form the situation. A detailed description of all these operations seems to involve insurmountable difficulties, but it is not essential for the formulation of a microscopic theory. For instance, the identification of the objects which form a situation should consist of successive approximations, starting from the identification of some macroscopic objects which every situation must contain. This first step is a problem of macroscopic physics, which does not need detailed discussion in our context.
(d) We assume that it is possible to perform operations which permit us to choose an integer belonging to the set

$$\Delta_p = \{0, 1, \ldots, p-1\} \tag{2.1}$$

† Standards of mass, length, etc. are not necessary, as nature itself provides natural units. If we disregard violations of charge conjugation symmetry, a specimen of positive charge is necessary in order to distinguish between charges of different sign.

in a random way, according to the given probabilities $a_0, \ldots, a_{p-1}$. Of course, we must have

$$a_r \geqslant 0, \qquad r = 0, \ldots, p-1 \tag{2.2}$$

$$\sum_{r=0}^{p-1} a_r = 1 \tag{2.3}$$

A procedure may contain prescriptions for physical operations which can possibly be interpreted as random choices, but can produce physical effects in many uncontrollable ways. On the contrary, the idealised random choices we are considering can have physical consequences only through their outcome and do not interfere materially with the other physical operations. We call them 'formal' random choices. Our assumptions are justified if we require that the formal random choices are performed long before the other physical operations. They can be obtained by means of classical devices and the details of the physical operations involved are irrelevant. Therefore, we shall consider as equal two procedures which differ from the practical realisation of the formal random choices.

(e) The prescriptions which define a procedure may contain conditional sentences, which prescribe a given operation only if a preceding operation has given a certain result, for instance if a formal random choice has given a certain outcome. If an operation is conditioned by an event which is logically impossible, or is generated by a formal random choice with zero probability, the document obtained from the original one by eliminating the corresponding prescriptions defines the same procedure.

The operation of 'weighted mean' defined in Giles (1970) can easily be extended to a finite set of procedures of the same type. We indicate by $\mathscr{G}$ the set of the procedures of a given kind treated by a given theory. Given p procedures $A^{(0)}, \ldots, A^{(p-1)}$ belonging to $\mathscr{G}$ and the real numbers $a_0, \ldots, a_{p-1}$ satisfying conditions (2.2) and (2.3), we can define the new procedure

$$A = \sum_{r=0}^{p-1} a_r A^{(r)} = a_0 A^{(0)} + \ldots + a_{p-1} A^{(p-1)} \in \mathscr{G} \tag{2.4}$$

by means of the following prescription.

Prescription: Choose the integer $r \in \Delta_p$, by means of a formal random choice, according to the probabilities $a_0, \ldots, a_{p-1}$. Then perform the procedure $A^{(r)}$. It is clear that in this way the required aims are attained.

It follows from the remarks given that a different ordering of the procedures $A^{(r)}$ and of the coefficients a_r is irrelevant for the definition of procedure (2.4). It is also clear that, if for a given r we have $a_r = 0$, the corresponding term can be eliminated in expression (2.4) without altering its meaning. For $p = 1$, we have:

$$1A = A \tag{2.5}$$

We say that a procedure is 'simple' if it does not contain formal random choices. Given a procedure A, all the formal random choices contained in it can be replaced, using the rules of probability theory, by a single random choice. The procedure A can then be written in the form (2.4), where the coefficients a_r are positive and the procedures $A^{(r)}$ are simple and each one different from the other. This decomposition is unique (up to the order of the various terms).

It follows that the set $\mathscr{G}$ can be considered as a convex set in the linear space of all the formal finite linear combinations of simple procedures belonging to $\mathscr{G}$. The operation of weighted mean, physically defined above, has the usual mathematical meaning and the usual formal properties. In other words, one can work with the expressions of type (2.4) by means of the usual rules of linear algebra. It is convenient to consider the affine manifold $\hat{\mathscr{G}}$ generated by the convex set $\mathscr{G}$. The affine space $\hat{\mathscr{G}}$ contains all the affine combinations of elements of $\mathscr{G}$, but only affine combinations with positive coefficients can be physically interpreted as weighted means.

We now consider some special kinds of procedures. By 'measurement procedure' we mean a procedure, the aim of which is to choose a 'result' in a given set of possible results. All the remarks given above are particularly valid for this kind of procedure. In order to satisfy remark (b), it is sufficient to give a last prescription, specifying the result to be chosen when some of the other prescriptions cannot be accomplished due to some unexpected difficulty.† The following additional remarks are also essential.

(f) An analysis of the actual experimental procedures shows that the set of possible results is necessarily finite. We call $\mathscr{E}_n$ the set of the measurement procedures with n possible results considered by our theory. We assume that the set of possible results has the form Δ_n (see equation (2.1)). The set $\mathscr{E}_1$ is composed of all the trivial measurement procedures which give '*a priori*' the only possible result, namely 0. As explained above, the set $\mathscr{E}_n$ can be considered as a convex set contained in an affine space $\hat{\mathscr{E}}_n$.

(g) A measurement procedure may require some mathematical operations which transform some preliminary results into the final result. Of course, in practice these mathematical operations are performed by means of physical operations; however, in analogy with remark (d), given above, we assume that they have no relevant physical effect and that they do not interfere with other physical operations. This idealisation is justified assuming that these mathematical operations are performed a long time after the other physical operations. The details of the physical operations required are irrelevant for our purposes.

† The measurement procedures described in Giles (1970) can produce, besides the two normal possible outcomes, the signal 'experiment void'. According to our conventions, we consider these three possible results on an equal footing and we assign these measurement procedures to the set $\mathscr{E}_3$.

Note, however, that the physical objects which carry the information which forms the result of a measurement procedure are by no means irrelevant in all steps of the procedure itself. They become irrelevant only when they assume a clearly macroscopic nature, as an electric signal in a computer or a sheet of printed paper.

Besides the weighted mean, we shall define another statistical relation between measurement procedures. We consider the set $\mathscr{M}_{mn}$ composed of all the real matrices φ_{ik} with m rows and n columns which satisfy the conditions:

$$\varphi_{ik} \geqslant 0, \qquad i = 0, \ldots, m-1, \qquad k = 0, \ldots, n-1 \tag{2.6}$$

$$\sum_{i=0}^{m-1} \varphi_{ik} = 1, \qquad k = 0, \ldots, n-1 \tag{2.7}$$

A matrix with these properties will be called a 'probability matrix'. $\mathscr{M}_{mn}$ is a convex compact set in the $(m-1)\,n$ dimensional affine space $\hat{\mathscr{M}}_{mn}$ of all the real $m \times n$ matrices which satisfy condition (2.7). Moreover, we have:

$$\varphi\varphi' \in \mathscr{M}_{mn} \qquad \text{if} \qquad \varphi \in \mathscr{M}_{ms}, \qquad \varphi' \in \mathscr{M}_{sn} \tag{2.8}$$

If $\varphi \in \mathscr{M}_{mn}$ and $A \in \mathscr{E}_n$, we can define the new measurement procedure

$$\varphi A \in \mathscr{E}_m \tag{2.9}$$

by means of the following prescription.

Prescription: For each integer $k \in \Delta_n$ we choose, by means of a formal random choice, an integer $i(k) \in \Delta_m$ using the probabilities φ_{ik}. We then perform the measurement procedure A. If the result is k, we choose, as the result of the measurement procedure φA, the integer $i(k)$.

Note that this prescription is in agreement with remark (d). We call φA a 'statistical alteration' of the measurement procedure A. This is a generalisation of the usual definition of a function of an observable (von Neumann, 1932).

We indicate by I the element of $\mathscr{E}_1$ defined by the simple prescription: 'the result is 0'. A measurement procedure of the kind

$$\begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} I \in \mathscr{E}_n \tag{2.10}$$

is simply a random choice of the result according to the probabilities $a_0, \ldots, a_{n-1}$. In particular, we shall consider the measurement procedures

$$O = \begin{pmatrix} 1 \\ 0 \end{pmatrix} I, \qquad U = \begin{pmatrix} 0 \\ 1 \end{pmatrix} I \tag{2.11}$$

belonging to $\mathscr{E}_2$, which give, '*a priori*', the result 0 and 1 respectively.

It follows from remarks (d) and (g) and from the rules of probability theory that the operation of statistical alteration has the following properties

$$\delta A = A \tag{2.12}$$

where δ is the unit matrix with the appropriate dimension,

$$\varphi(\varphi' A) = (\varphi\varphi')A \tag{2.13}$$

$$(\sum_r a_r \varphi^{(r)})(\sum_s b_s A^{(s)}) = \sum_{rs} a_r b_s (\varphi^{(r)} A^{(s)}) \tag{2.14}$$

In fact, the measurement procedures connected by the equality sign differ only by the practical details of some mathematical operations or of some formal random choices. In order to save space, we have not written the conditions of validity explicitly; we understand that these equalities hold whenever the operations involved are well defined.

We now define another kind of procedure, which we call 'transformation procedures'. Their aim is to construct a situation. We remember that a situation is a set of material objects which indicate the space-time conditions in which the operations prescribed by a procedure have to be performed. We call $\mathscr{F}$ the set of the transformation procedures considered by our theory. $\mathscr{F}$ can be considered as a convex set in the affine space $\hat{\mathscr{F}}$.

The remarks (a)–(e), given above, also hold for the transformation procedures. In particular, the prescriptions which define a transformation procedure necessarily refer to some pre-existent material objects which form another situation. Following the prescriptions of the transformation procedure, one transforms one situation into another new situation. We see that only relations between situations, and not the situations themselves, have a direct operational meaning. The concept of situation will not appear in our mathematical formalism.

According to remark (b), the prescriptions defining a transformation procedure must be formulated in such a way that, following them, one always succeeds in building a situation. This requirement is, necessarily, somewhat obscure, as we have not yet decided explicitly which kind of material objects can form a situation. These details have to be specified carefully in each particular theory.

If $A \in \mathscr{E}_n$ and $F \in \mathscr{F}$, we can define the measurement procedure $AF \in \mathscr{E}_n$, called the 'composition' of A and F by means of the following prescription.

Prescription: In order to perform the procedure AF in a given situation, one has to perform the operations of F using the given situation and the operations of A using the situation constructed by F. The result of A obtained in this way is the result of AF.

We remember that, according to remark (b) given above, the procedure A must contain some rules prescribing alternative operations whenever a prescription of A and a prescription of F are in contradiction. This requirement gives rise to difficult problems, but it seems to be unavoidable. It should be clear, after what we have said, that one cannot avoid these problems requiring that every operation of A has to be performed after the operations of F.

In a perfectly similar way, if $G, F \in \mathscr{F}$, we can define their composition $GF \in \mathscr{F}$.

Using remarks (d) and (g), we can easily justify the following equalities, which hold whenever the operations involved are meaningful:

$$(AF)G = A(FG) \tag{2.15}$$

$$(FG)H = F(GH) \tag{2.16}$$

$$\Big(\sum_r a_r A^{(r)}\Big)\Big(\sum_s b_s F^{(s)}\Big) = \sum_{rs} a_r b_s (A^{(r)} F^{(s)}) \tag{2.17}$$

$$\Big(\sum_r a_r F^{(r)}\Big)\Big(\sum_s b_s G^{(s)}\Big) = \sum_{rs} a_r b_s (F^{(r)} G^{(s)}) \tag{2.18}$$

$$(\varphi A)F = \varphi(AF) \tag{2.19}$$

The letter A, with or without indices, denotes measurement procedures and the letters F, G, H, with or without indices, denote transformation procedures.

For the mathematical development of the concepts introduced above, it is useful to remark that the operation of statistical alteration can be extended to an affine mapping of $\hat{\mathscr{M}}_{mn} \times \hat{\mathscr{E}}_n$ into $\hat{\mathscr{F}}_m$. In a similar way, the operations of composition can be extended to affine mappings of $\hat{\mathscr{E}}_n \times \hat{\mathscr{F}}$ into $\hat{\mathscr{E}}_n$ and of $\hat{\mathscr{F}} \times \hat{\mathscr{F}}$ into $\hat{\mathscr{F}}$. These extensions can be obtained by means of the proposition proved in the Appendix.

The concepts of a frame of reference and of a transformation of the Poincaré group,† which appear in relativistic theories, can be considered as idealisations of the concepts of situation and of transformation procedure. We think that an operational approach to the structure of space-time should be based on the analysis of the transformation procedures.

We remark, however, that a transformation procedure is not just a generalised change of reference. It implies physical operations which may affect the material objects under investigation. We shall see in Section 5 that the transformation procedures can also be used to generalise and replace the preparation procedures which should define the physical states.

3. *An Order Relation Between Measurement Procedures of* $\mathscr{E}_2$

The formalism described in the preceding section can be considered as a scheme for the classification of experimental techniques. Now we want to complete this scheme in order to include the formulation of the statistical physical laws which connect the results of the experiments. We achieve this end by introducing a relation, denoted by the sign $\leqslant$, between the elements of the set $\mathscr{E}_2$. We have to explain how the statement $A \leqslant B$ can be tested experimentally.

As we have anticipated in the Introduction, we need the following assumption.

† More exactly, one should consider the semigroup generated by the homogeneous orthochronous Lorentz transformations and by the space-time translations belonging to the future cone. Only the elements of this semigroup have a direct operational meaning in macroscopic physics.

Assumption: Given a finite set $\mathscr{H}$ of procedures and an integer N, one can build a set of N situations in such a way that it is possible to perform, in each of the N situations, an arbitrarily chosen procedure of the set $\mathscr{H}$. This means that, putting together all the prescriptions corresponding to the procedures to be performed in the N situations, we obtain a set of prescriptions free of ambiguities and contradictions.

Practically, one has to choose the N situations sufficiently separated in space-time. Note that the separation required depends on the nature of the procedures contained in the set $\mathscr{H}$. A set of situations which satisfies these conditions will be called an 'ensemble of situations' suitable for the set $\mathscr{H}$ of procedures.

Of course, in order to build an ensemble of situations, one has to perform certain operations following some set of prescriptions which form a new kind of procedure. As these operations have a classical macroscopic nature, we shall not discuss this kind of procedure further. It is important to remark that two ensembles of situations built by means of the same procedure cannot be considered as equal or equivalent in any sense.

Note that we are not assuming that the result of a measurement procedure is not influenced by the fact that other $(N - 1)$ procedures are performed. It is rather difficult to give a clear operational meaning to this assumption and it does not seem to be necessary for our purpose. We think, however, that this point should be analysed with more detail.

It follows from our definition that we can always enlarge the set $\mathscr{H}$ in such a way that we have

$$[\sum_r a_r A^{(r)} \in \mathscr{H} \quad \text{and} \quad a_s > 0] \Rightarrow [A^{(s)} \in \mathscr{H}] \tag{3.1}$$

$$[\varphi A \in \mathscr{H}] \Rightarrow [A \in \mathscr{H}] \tag{3.2}$$

A finite number M of 'equivalent' ensembles of situations can be obtained considering one ensemble of situations and decomposing it into M parts in a perfectly random way. Of course, these random choices have to be performed much in advance.

If S is an ensemble of situations suitable for a set $\mathscr{H}$ containing the measurement procedure $A \in \mathscr{E}_n$, we can perform A in all the situations of S and we indicate by $P(A, S, i)$ the frequency of the result $i \in \Delta_n$. Of course, we have

$$P(A, S, i) \geqslant 0, \qquad i = 0, \ldots, n-1, \tag{3.3}$$

$$\sum_{i=0}^{n-1} P(A, S, i) = 1 \tag{3.4}$$

It is important to note that it would not be correct to consider the quantity $P(A, S, i)$, for fixed S and i, as a function of A defined in $\mathscr{E}_n$. In fact, the ensemble of situations S cannot be reproduced at will, and for a fixed S the quantity P can be measured only for one single measurement procedure A.

By applying the rules of probability theory to the random choices which have to be performed in order to construct equivalent ensembles of situations, or which are required by the prescriptions which define the measurement procedures, we get the following results. If S, S', $S^{(0)}$, $S^{(1)}$, . . . are equivalent ensembles of situations suitable for the set $\mathscr{H}$ containing the relevant procedures, we have (whenever these expressions are meaningful)

$$P(A, S, i) \approx P(A, S', i) \tag{3.5}$$

$$P(\sum_r a_r A^{(r)}, S, i) \approx \sum_r a_r P(A^{(r)}, S^{(r)}, i) \tag{3.6}$$

$$P(\varphi A, S, i) \approx \sum_k \varphi_{ik} P(A, S', k) \tag{3.7}$$

Here, and in the following, we indicate by $\approx$ and $\lesssim$, respectively, the equality and the inequality of two real quantities within the expected statistical fluctuations.

In order to test the relation $A \leqslant B$ between the measurement procedures A and B, both belonging to $\mathscr{E}_2$, we build two equivalent ensembles of situations S and S', both suitable for a set $\mathscr{H}$ of measurement procedures containing A and B. We then perform A in all the situations of S, and B in all the situations of S', and we require that

$$P(A, S, 1) \lesssim P(B, S', 1) \tag{3.8}$$

Of course, this test has to be repeated for many pairs of equivalent ensembles of situations, as large and various as possible.

We remark (Popper, 1935, 1959) that, as in any other physical law, the relation $A \leqslant B$ cannot be proved experimentally. It can be disproved in a statistical sense if, performing a test of the kind described above, the inequality (3.8) is contradicted well beyond the expected statistical fluctuations. We find here the unavoidable conceptual difficulties connected with testing a statistical law.

We now have to discuss a serious objection to the approach described above. Assume that, using two equivalent ensembles of situations S and S', we have found that the inequality (3.8) is false. As the ensembles of situations are not reproducible at will, another physicist has no possibility of performing an independent test on our result. This result can be attributed to some mistake or to some exceptionally large statistical fluctuation and cannot be considered as an objective disproof of the statement $A \leqslant B$.

It follows that our theory has a rather unsatisfactory degree of objectivity. The considerations which follow give a partial solution to this problem. We shall show in Section 5 that in the usual theories, based on the concept of state, the problem is hidden but no less serious.

If A is a measurement procedure, F is a transformation procedure and S is an ensemble of situations suitable for the measurement procedure AF, applying F to all the situations of S we get a new ensemble of situations which we indicate by FS. The ensemble FS is suitable for the measurement procedure A and we can write

$$P(A, FS, i) = P(AF, S, i) \tag{3.9}$$

because these two quantities are measured by means of the same set of experiments.

It follows that if $A \leqslant B$ and S, S' are equivalent ensembles of situations we have

$$P(AF, S, 1) \lesssim P(BF, S', 1) \tag{3.10}$$

Therefore we can write:

$$[A \leqslant B] \Rightarrow [AF \leqslant BF], \qquad A, B \in \mathscr{E}_2 \tag{3.11}$$

The most effective way of disproving the relation $A \leqslant B$ is to use an ensemble of situations expressly prepared by means of operations which are believed to be particularly suitable for this aim. For instance, one can use an ensemble of the form FS where F is a suitably chosen transformation procedure. In this way, one tests the relation $A \leqslant B$ through its consequence $AF \leqslant BF$.

An objective refutation of the relation $A \leqslant B$ can be given by finding a transformation procedure F such that the inequality (3.10) is clearly false for any pair of equivalent ensembles of situations sufficiently numerous for disregarding the statistical fluctuations.

This possibility of objective disproof increases the degree of objectivity of the theory, but the situation is not yet completely satisfactory. In fact, we cannot exclude the embarrassing case in which the relation $A \leqslant B$ has been disproved occasionally but no objective disproof has been found.

Summarising, a theory of the type we are considering is given by:

(a) The sets $\mathscr{E}_n$ and $\mathscr{F}$, the elements of which are physically interpreted as procedures of the kinds described in the preceding section.
(b) The operations of weighted mean, statistical alteration and composition, which have the physical interpretation and the properties explained in the preceding section.
(c) The relation $\leqslant$ defined in the set $\mathscr{E}_2$, which is the only kind of statement of the theory which is subject to experimental tests, as explained above.

A physical theory of this kind is valid if the physical interpretations of all these concepts are operationally well defined and if the relation $\leqslant$ is not disproved by experiment.

We remark that, in this framework, the time evolution laws can be expressed by means of relations of the form $AF \leqslant B$, where A and B belong to $\mathscr{E}_2$ and F is a transformation procedure which can be interpreted as a translation in time.

Physical theories are subject to a continuous evolution. In the scheme described above, improvements of the following kinds may be introduced.

(a) The sets $\mathscr{E}_n$ and $\mathscr{F}$ are enlarged in order to take into account the progress of the experimental techniques.
(b) Some relations of the type $A \leqslant B$ have to be eliminated, because they have been contradicted by experiments.
(c) Some new relations of the kind $A \leqslant B$ are assumed and, after having been successfully submitted to a large number of suitably chosen experimental tests, they are provisionally accepted as true. In this way, one increases the physical content of the theory.

We stress that the relation $\leqslant$ cannot be induced once and for ever from experiments, but it has to be assumed as a somewhat arbitrary hypothesis to be controlled experimentally.

It should be clear that the current physical theories have not the idealised structure described above. In particular, as we shall see in Section 7, a detailed list of the procedures considered by the theory is never available. Also the historical development of physics cannot be understood correctly by means of the oversimplified scheme given above. In general, the relation $\leqslant$ is described mathematically in terms of other mathematical structures which have a less direct operational meaning, but can more easily be defined and treated mathematically. We say that they form the 'mathematical model' of the theory. Often one can assign an intuitive physical meaning to some terms appearing in the mathematical model and, in this way, one gets a physical model. In many cases the change in the relation $\leqslant$ which corresponds to an improvement of the theory implies a radical change of the mathematical model. This essential aspect of the development of the physical theories is not discussed here.

In the formulation of an hypothesis concerning the relation $\leqslant$ it is convenient to take into account some consequences of the laws of probability theory which are contained in equations (3.5)-(3.7). We shall always assume that the relation $\leqslant$ has the following seven properties:

$$O \leqslant A \leqslant U, \qquad A \in \mathscr{E}_2 \tag{3.12}$$

$$\binom{1}{0}E \leqslant O, \qquad U \leqslant \binom{0}{1}E, \qquad E \in \mathscr{E}_1 \tag{3.13}$$

$$A \leqslant A \tag{3.14}$$

$$[A^{(r)} \leqslant B^{(r)}, r = 0, \ldots, n-1] \Rightarrow [\sum_r a_r A^{(r)} \leqslant \sum_r a_r B^{(r)}] \tag{3.15}$$

$$[A \leqslant C \text{ and } C \leqslant B] \Rightarrow [A \leqslant B] \tag{3.16}$$

$$[0 < a \leqslant 1 \text{ and } C \leqslant D \text{ and } aA + (1-a)D \leqslant aB + (1-a)C] \Rightarrow [A \leqslant B] \tag{3.17}$$

$$[A \leqslant (1-a)B + aC \text{ for } 0 < a < 1] \Rightarrow [A \leqslant B] \tag{3.18}$$

It is understood that these formulae hold whenever the expressions contained by them are well defined.

We easily see that the first two equations can never be disproved experimentally. Equation (3.14) is justified by equation (3.5). In order to justify equation (3.15), we remark that when we test the relation in the second square bracket, using the equivalent ensembles of situations S and S' we can also consider the $2n$ ensembles of situations $S^{(r)}$ and $S'^{(r)}$ equivalent to S and S'. From equation (3.1) it follows that all these ensembles of situations are suitable for all the measurement procedures $A^{(r)}$ and $B^{(r)}$, therefore we can use them to test the relations $A^{(r)} \leqslant B^{(r)}$. If these relations are not contradicted we obtain

$$P(A^{(r)}, S^{(r)}, 1) \lesssim P(B^{(r)}, S'^{(r)}, 1), \qquad r = 0, \ldots, n-1 \tag{3.19}$$

Using equation (3.6) twice we finally obtain

$$P(\sum_r a_r A^{(r)}, S, 1) \lesssim P(\sum_r a_r B^{(r)}, S', 1) \tag{3.20}$$

in accordance with the relation we wanted to test.

The assumptions (3.16)-(3.18) are more difficult to justify. In fact, when we test the relation $A \leqslant B$ by means of the equivalent ensembles of situations S and S', we can also build other ensembles of situations equivalent to S and S', but we cannot be sure that they are suitable for the measurement procedures C and D. If we disregard this difficulty, the assumptions (3.16)-(3.18) can be justified in the same way as the preceding one. A complete clarification of this point would require a deeper analysis of the concept of ensemble of situations.

Equations (3.14) and (3.16) mean that $\leqslant$ is an order relation (in general, non-antisymmetric). Equations (3.15) and (3.17) can be used to extend this order relation to the whole affine space $\hat{\mathscr{E}}_2$. It is convenient to transform $\hat{\mathscr{E}}_2$ into a linear space by choosing the origin in the point O. We can then use the following proposition:

Proposition 1: If $\mathscr{E}_2$ is a convex set in a vector space $\hat{\mathscr{E}}_2$ and $\leqslant$ is an order relation defined in $\mathscr{E}_2$ with the properties (3.15) and (3.17), it is always possible to define an order relation on $\hat{\mathscr{E}}_2$ (compatible with its structure of vector space (Jameson, 1970)) which, when restricted to $\mathscr{E}_2$, coincides with the order relation previously given.

Proof: We consider the cone $\mathscr{E}^+$ composed of all the vectors of the form

$$C = a(B - A), \qquad a \geqslant 0, \qquad A, B \in \mathscr{E}_2 \qquad A \leqslant B \tag{3.21}$$

This cone is convex. In fact, if we consider another element

$$C' = a'(B' - A'), \qquad a' \geqslant 0, \qquad A', B' \in \mathscr{E}_2, \qquad A' \leqslant B' \tag{3.22}$$

of $\mathscr{E}^+$, we have

$$C + C' = (a + a')(B'' - A'') \tag{3.23}$$

where

$$\begin{cases} A'' = \dfrac{a}{a+a'} A + \dfrac{a'}{a+a'} A' \\ B'' = \dfrac{a}{a+a'} B + \dfrac{a'}{a+a'} B' \end{cases} \tag{3.24}$$

From equation (3.15) we see that $A'' \leqslant B''$ and therefore we have $C + C' \in \mathscr{E}^+$.

We show that if $A, B \in \mathscr{E}_2$, we have

$$[B - A \in \mathscr{E}^+] \Leftrightarrow [A \leqslant B] \tag{3.25}$$

It is clear that the first statement follows from the second. If the second statement is true, we can write

$$B - A = a(B' - A'), \qquad a \geqslant 0, \qquad A', B' \in \mathscr{E}_2, \qquad A' \leqslant B' \tag{3.26}$$

and therefore

$$\frac{1}{a+1} A + \frac{a}{a+1} B' = \frac{1}{a+1} B + \frac{a}{a+1} A' \tag{3.27}$$

and, using equation (3.17), we get $A \leqslant B$.

In conclusion, we can use equation (3.25) in order to define the order relation in the whole space $\hat{\mathscr{E}}_2$ and the proposition is proved.

From the order relation $\leqslant$ we can derive an equivalence relation in the usual way:

$$[A \equiv B] \Leftrightarrow [A \leqslant B \text{ and } B \leqslant A] \tag{3.28}$$

This equivalence relation is compatible with the linear structure of $\hat{\mathscr{E}}_2$.

From equations (3.12) and (3.13) we get

$$\binom{1}{0} E \equiv O, \qquad \binom{0}{1} E \equiv U, \qquad E \in \mathscr{E}_1 \tag{3.29}$$

It is easy to show that these formulae also hold for $E \in \hat{\mathscr{E}}_1$. If $A \in \hat{\mathscr{E}}_2$ and we put

$$E = (1 \;\; 1) A \in \hat{\mathscr{E}}_1 \tag{3.30}$$

we have

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A + \tfrac{1}{2} A = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} E + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} E \equiv \tfrac{1}{2} U + \tfrac{1}{2} O \tag{3.31}$$

and with our choice of the origin in $\hat{\mathscr{E}}_2$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \equiv U - A \tag{3.32}$$

As the linear space $\hat{\mathscr{E}}_2$ is generated by the convex set $\mathscr{E}_2$, every element $C \in \hat{\mathscr{E}}_2$ can be written in the form

$$C = aA - bB, \qquad a, b \geqslant 0, \qquad A, B \in \mathscr{E}_2 \tag{3.33}$$

It follows that

$$-bU \leqslant C \leqslant aU \tag{3.34}$$

This means that the element U is an order unit (Jameson, 1970).

From equation (3.18) it follows that the ordering of $\hat{\mathscr{E}}_2$ is archimedean, namely we have (Jameson, 1970):

$$[C \leqslant aU \text{ for } a > 0] \Rightarrow [C \leqslant O]. \tag{3.35}$$

In fact, if we put

$$C = c(A - B), \qquad c > 0, \qquad A, B \in \mathscr{E}_2, \qquad 2B \leqslant U \tag{3.36}$$

from the first statement of equation (3.35) we obtain

$$A \leqslant \frac{a}{c} U + B \leqslant 2\frac{a}{c} U + \left(1 - 2\frac{a}{c}\right) B \tag{3.37}$$

and from equation (3.18) we get $A \leqslant B$ and therefore $C \leqslant O$.

In order to define an equivalence relation in the spaces $\hat{\mathscr{E}}_n$, with $n \neq 2$, we define the probability matrices $\psi^{(nr)} \in \mathscr{M}_{2n}$ in the following way:

$$\psi_{1k}^{(nr)} = \delta_{rk}, \qquad \psi_{0k}^{(nr)} = 1 - \delta_{rk}, \qquad k = 0, \ldots, n-1 \tag{3.38}$$

Then if A and B belong to $\hat{\mathscr{E}}_n$, we define the equivalence relation $A \equiv B$ as follows:

$$[A \equiv B] \Leftrightarrow [\psi^{(nr)} A \equiv \psi^{(nr)} B, \qquad r = 0, \ldots, n-1] \tag{3.39}$$

Using equation (3.32) we see that this formula is also true for $n = 2$. From equation (3.29) we see that, according to our definition, all the elements of $\hat{\mathscr{E}}_1$ are equivalent to one another.

Using equation (2.14), we easily see that this equivalence relation is compatible with the structure of affine space of $\hat{\mathscr{E}}_n$. Moreover, using the identity

$$\psi^{(ms)} \varphi = \sum_{r=0}^{n-1} \varphi_{sr}\, \psi^{(nr)} + (1 - \sum_{r=0}^{n-1} \varphi_{sr}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ldots 1), \qquad \varphi \in \hat{\mathscr{M}}_{mn} \tag{3.40}$$

we can easily show that

$$[A \equiv B] \Rightarrow [\varphi A \equiv \varphi B] \tag{3.41}$$

We now consider equation (3.11). One can easily show that it can be extended to the case in which $A, B \in \hat{\mathscr{E}}_2$. From this equation it follows:

$$[A \equiv B] \Rightarrow [AF \equiv BF], \qquad F \in \mathscr{F} \tag{3.42}$$

Using equations (2.19) and (3.39), we see that this equation holds also for $A, B \in \hat{\mathscr{E}}_n$.

4. *The Mathematical Structure of Measurements*

In the present section we commence a mathematical investigation of the concepts and of the assumptions discussed in Sections 2 and 3.

As the equivalence relation introduced in the preceding section is compatible with the affine structure of the spaces $\hat{\mathscr{E}}_n$, we can introduce the quotient affine spaces $\hat{\boldsymbol{\mathscr{E}}}_n$. An element of $\hat{\boldsymbol{\mathscr{E}}}_n$ is a set of equivalent elements of $\hat{\mathscr{E}}_n$. If the equivalence class $\boldsymbol{A} \in \hat{\boldsymbol{\mathscr{E}}}_n$ contains at least a measurement procedure $A \in \mathscr{E}_n$, we say that it is a 'measurement'. We indicate by $\boldsymbol{\mathscr{E}}_n$ the set of all the measurements contained in $\hat{\boldsymbol{\mathscr{E}}}_n$. $\boldsymbol{\mathscr{E}}_n$ is a convex set in the affine space $\hat{\boldsymbol{\mathscr{E}}}_n$. We can also consider a measurement as a set of equivalent measurement procedures. The measurements of $\boldsymbol{\mathscr{E}}_2$, which play a special role in the formalism we are describing, will also be called 'tests' (Giles, 1970).

The space $\hat{\boldsymbol{\mathscr{E}}}_2$ also has a structure of ordered linear space and it is easy to show that the ordering is antisymmetric and archimedean (Jameson, 1970). We indicate by $\boldsymbol{O}$ and $\boldsymbol{U}$ the tests which contain O and U respectively. They contain all the measurement procedures of $\mathscr{E}_2$ which always (but not necessarily '*a priori*') give the result 0 and 1 respectively. $\boldsymbol{O}$ is the zero element of the space $\hat{\boldsymbol{\mathscr{E}}}_2$ and $\boldsymbol{U}$ is an order unit of the same space. The space $\hat{\boldsymbol{\mathscr{E}}}_1$ contains only one element which we indicate by $\boldsymbol{I}$.

Equation (3.41) means that the equivalence relation is compatible with the operation of statistical alteration. It follows that we can define, in a natural way, an affine mapping of $\mathscr{M}_{mn} \times \hat{\boldsymbol{\mathscr{E}}}_n$ into the space $\hat{\boldsymbol{\mathscr{E}}}_m$, which we again call statistical alteration. If $\boldsymbol{A} \in \boldsymbol{\mathscr{E}}_n$ and $\varphi \in \mathscr{M}_{mn}$, we have $\varphi \boldsymbol{A} \in \boldsymbol{\mathscr{E}}_m$. In particular we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{I} = \boldsymbol{O}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boldsymbol{I} = \boldsymbol{U} \tag{4.1}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{A} = \boldsymbol{U} - \boldsymbol{A}, \qquad \boldsymbol{A} \in \hat{\boldsymbol{\mathscr{E}}}_2 \tag{4.2}$$

It is useful to summarise the geometric properties of the set $\boldsymbol{\mathscr{E}}_2$ in the following proposition:

Proposition 2: $\boldsymbol{\mathscr{E}}_2$ is a convex set in the ordered linear space $\hat{\boldsymbol{\mathscr{E}}}_2$. It contains the elements $\boldsymbol{O}$ and $\boldsymbol{U}$, it is symmetric with respect to the reflection

$$\boldsymbol{A} \to \boldsymbol{U} - \boldsymbol{A} \tag{4.3}$$

and has the property

$$\boldsymbol{O} \leqslant \boldsymbol{\mathscr{E}}_2 \leqslant \boldsymbol{U} \tag{4.4}$$

From equation (3.39) we have:

$$[\boldsymbol{A} = \boldsymbol{B}] \Leftrightarrow [\psi^{(nr)}\boldsymbol{A} = \psi^{(nr)}\boldsymbol{B},\ r = 0, \ldots, n-1] \qquad \boldsymbol{A}, \boldsymbol{B} \in \hat{\boldsymbol{\mathscr{E}}}_n \tag{4.5}$$

In other words, the element $A \in \hat{\mathscr{E}}_n$ is uniquely determined by the n elements

$$A_r = \psi^{(nr)} A \in \hat{\mathscr{E}}_2, \qquad r = 0, \ldots, n-1 \tag{4.6}$$

which we call the 'components' of A. We shall use the notation:

$$A = [A_0, A_1, \ldots, A_{n-1}] \in \hat{\mathscr{E}}_n \tag{4.7}$$

In particular, we have:

$$A = [U - A, A] \in \hat{\mathscr{E}}_2 \tag{4.8}$$

$$I = [U] \tag{4.9}$$

The operations of weighted mean and of statistical alteration can be expressed in terms of components in the following way.

Proposition 3: We have:

$$\sum_k a_k [A_0^{(k)}, \ldots, A_{n-1}^{(k)}] = [A_0, \ldots, A_{n-1}] \tag{4.10}$$

where

$$A_r = \sum_k a_k A_r^{(k)}, \qquad r = 0, \ldots, n-1 \tag{4.11}$$

and

$$\varphi[A_0, \ldots, A_{n-1}] = [B_0, \ldots, B_{m-1}], \qquad \varphi \in \hat{\mathscr{M}}_{mn} \tag{4.12}$$

where

$$B_s = \sum_r \varphi_{sr} A_r \tag{4.13}$$

Proof: The first result follows immediately from definition (4.6) of the component. Using the identity (3.40), we get the formula:

$$B_s = \psi^{(ms)} \varphi A = \sum_r \varphi_{sr} A_r + (1 - \sum_r \varphi_{sr}) O \tag{4.14}$$

which coincides, with equation (4.13), with our choice of the origin in $\hat{\mathscr{E}}_2$.

The following proposition shows that the structure of all the spaces $\hat{\mathscr{E}}_n$ is uniquely determined by the structure of the space $\hat{\mathscr{E}}_2$.

Proposition 4: The equation (4.6) permits us to identify the space $\hat{\mathscr{E}}_n$ with the affine space of all the sequences $[A_0, \ldots, A_{n-1}]$ of elements of $\hat{\mathscr{E}}_2$ which satisfy the condition

$$\sum_{r=0}^{n-1} A_r = U \tag{4.15}$$

Proof: Condition (4.15) follows from the definition (4.6) and from the identity

$$\sum_{r=0}^{n-1} \frac{1}{n} \psi^{(nr)} = \frac{1}{n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \ldots 1) + \frac{n-1}{n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ldots 1) \tag{4.16}$$

In order to prove that all sequences of this kind define an element of $\hat{\mathscr{E}}_n$, we consider the following elements of $\hat{\mathscr{E}}_n$:

$$\begin{cases} B_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} I = [U, O, \ldots, O] \\ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 0 \end{pmatrix} A_1 = [U - A_1, A_1, O, \ldots, O] \\ \ldots \\ B_{n-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} A_{n-1} = [U - A_{n-1}, O, \ldots, O, A_{n-1}] \end{cases} \tag{4.17}$$

Then, using equation (4.15), we obtain:

$$[A_0, \ldots, A_{n-1}] = (2-n) B_0 + B_1 + \ldots + B_{n-1} \in \hat{\mathscr{E}}_n \tag{4.18}$$

It is important to note that the sets of measurements $\mathscr{E}_n$ are not determined univocally by the set $\mathscr{E}_2$. Their structure contains independent physical information. The following proposition gives a limitation to the extension of the sets $\mathscr{E}_n$.

Proposition 5: The components A_r of a measurement $A \in \mathscr{E}_n$ satisfy the condition

$$\sum_r a_r A_r \in \mathscr{E}_2 \tag{4.19}$$

for any choice of the numbers a_r such that

$$O \leqslant a_r \leqslant 1, \qquad r = 0, \ldots, n-1 \tag{4.20}$$

We remark that, as $\mathscr{E}_2$ is convex, it is sufficient to impose condition (4.19) with the restriction that the coefficients a_r can take only the values 0 and 1.

Proof: We just have to apply Proposition 3 to the relation

$$\begin{pmatrix} 1 - a_0 \ldots 1 - a_{n-1} \\ a_0 \ldots \quad a_{n-1} \end{pmatrix} A \in \mathscr{E}_2 \tag{4.21}$$

Summarising, we have translated the relations between measurement procedures into relations between the corresponding measurements. In this way we obtain a separation of problems which correspond roughly to the distinction between experimental and theoretical physics. The first class of problems deals with the description of the measurement procedures and their assignment to the appropriate measurement; the second class of problems deals with the mathematical relations between measurements.

We have described these relations in terms of a mathematical structure, which we call the 'structure of measurements'. It is composed of the following elements:

(a) The ordered linear space $\hat{\mathscr{E}}_2$ with antisymmetric and archimedean ordering, with the positive cone $\mathscr{E}^+$ and with the order unit U. As explained in Propositions 3 and 4, starting from these elements one can construct the affine spaces $\hat{\mathscr{E}}_n$, with $n = 1, 3, 4, \ldots$. and define the operation of statistical alteration.

(b) The convex sets $\mathscr{E}_n$ which contain the measurements. The set $\mathscr{E}_2$ must have the properties required by Proposition 2. The other sets $\mathscr{E}_n$ must be chosen in such a way that the statistical alteration of a measurement is a measurement. In particular they must satisfy the condition of Proposition 5.

If the set $\mathscr{E}_2$ is given, a simple consistent way of defining the other sets $\mathscr{E}_n$ is to assign to $\mathscr{E}_n$ all the elements of $\hat{\mathscr{E}}_n$ which satisfy the condition of Proposition 5. It is easy to show that with this definition the operation of statistical alteration maps $\mathscr{M}_{mn} \times \mathscr{E}_n$ into $\mathscr{E}_m$. If the sets $\mathscr{E}_n$ are defined in this way, we say that the structure of measurements is 'full'.

If the ordered linear space $\hat{\mathscr{E}}_2$ is given, a simple consistent way of defining the set $\mathscr{E}_2$ is to put

$$\mathscr{E}_2 = \mathscr{E}^+ \cap (U - \mathscr{E}^+) \tag{4.22}$$

namely

$$[A \in \mathscr{E}_2] \Leftrightarrow [O \leqslant A \leqslant U] \tag{4.23}$$

If the set $\mathscr{E}_2$ is defined in this way, we say that the structure of measurements is 'conical'.

It is easy to show that if a structure of measurements is both full and conical, the sets $\mathscr{E}_n$ are defined by the conditions

$$A_r \geqslant O, \qquad r = 0, \ldots n-1 \tag{4.24}$$

In this case the structure of measurements is uniquely determined by the ordered linear space $\hat{\mathscr{E}}_2$ with order unit U.

There is no general physical justification for assuming that the structure of measurements is full or conical. However, the full conical structure of measurements generated by the ordered linear space $\hat{\mathscr{E}}_2$ can be obtained from the original structure of measurements introducing some new 'ideal' measurements which do not correspond to any known measurement procedure. The introduction of 'extended' structures of measurements of this kind will be discussed in Section 7.

In any case, a great deal of important information on physical theory is contained in the structure of the ordered linear space $\hat{\mathscr{E}}_2$. The approach to general quantum theory in terms of ordered linear space has been proposed and developed by several authors (Giles, 1970; Ludwig, 1970; Wick, Wightman & Wigner, 1974).

It is useful to introduce in the space $\hat{\mathscr{E}}_2$ the following order unit seminorm:

$$\|A\| = \inf[c: -cU \leqslant A \leqslant cU] \tag{4.25}$$

As the ordering of $\hat{\mathscr{E}}_2$ is antisymmetric and archimedean, this seminorm is indeed a norm and the positive cone $\mathscr{E}^+$ is closed with respect to the norm topology (Jameson, 1970).

In order to clarify the physical meaning of this norm, we consider two tests A and B containing the measurement procedures A and B. From the inequality

$$-cU \leqslant A - B \leqslant cU \tag{4.26}$$

it follows

$$\frac{c}{1+c} O + \frac{1}{1+c} A \leqslant \frac{c}{1+c} U + \frac{1}{1+c} B \tag{4.27}$$

$$\frac{c}{1+c} O + \frac{1}{1+c} B \leqslant \frac{c}{1+c} U + \frac{1}{1+c} A \tag{4.28}$$

If S and S' are two equivalent ensembles of situations, we get from equation (4.27)

$$\frac{1}{1+c} P(A, S, 1) \lesssim \frac{c}{1+c} + \frac{1}{1+c} P(B, S', 1) \tag{4.29}$$

Using also equation (4.28) and equation (4.25) we finally obtain

$$|P(A, S, 1) - P(B, S', 1)| \lesssim \|A - B\| \tag{4.30}$$

We see that it is very difficult to prove the inequivalence of the two measurement procedures A and B if $\|A - B\|$ is very small.

We can also introduce a topology in the spaces $\hat{\mathscr{E}}_n$ by means of the distance

$$d(A, B) = \sup_{O \leqslant r \leqslant n-1} \|A_r - B_r\| \tag{4.31}$$

One can easily see that with this topology the operation of statistical alteration is continuous.

We say that the structure of measurements is 'complete' if the space $\hat{\mathscr{E}}_2$ is complete and all the sets $\mathscr{E}_n$ are closed. In this case all the spaces $\hat{\mathscr{E}}_n$ and all the sets $\mathscr{E}_n$ are complete. One can always get a complete structure of measurements taking the completion of the space $\hat{\mathscr{E}}_2$ and the closure of the positive cone $\mathscr{E}^+$ and of the sets $\mathscr{E}_n$. Of course, in this completion procedure, one introduces new 'ideal' measurements which do not correspond to any known measurement procedure.

5. *The Mathematical Structure of Transformations*

In this section we continue the mathematical investigation initiated in the preceding section, taking the transformation procedures into account.

Equation (3.42) shows that the equivalence relation is compatible with the operation of composition. It follows that a transformation procedure $F \in \mathscr{F}$ defines (for each integer $n \geqslant 1$) an affine mapping of $\hat{\mathscr{E}}_n$ into itself which maps the set $\mathscr{E}_n$ into itself. We indicate these mappings by $\boldsymbol{F}$ and we use the notation

$$AF = A\boldsymbol{F} \in \hat{\mathscr{E}}_n, \qquad A \in \hat{\mathscr{E}}_n \tag{5.1}$$

From equation (2.19) we have:

$$(\varphi A)\boldsymbol{F} = \varphi(A\boldsymbol{F}) \tag{5.2}$$

and from equation (4.6) we obtain:

$$[A_0, \ldots, A_{n-1}]\boldsymbol{F} = [A_0\boldsymbol{F}, \ldots, A_{n-1}\boldsymbol{F}] \tag{5.3}$$

We see that the mapping $\boldsymbol{F}$ of $\hat{\mathscr{E}}_2$ into itself determines uniquely the mapping of $\hat{\mathscr{E}}_n$ into itself for arbitrary n.

The affine mappings of $\hat{\mathscr{E}}_2$ into itself which correspond to transformation procedures will be called 'transformations' and we indicate by $\boldsymbol{\mathscr{F}}$ the set of all the transformations of our theory. A transformation can also be considered as a set of 'equivalent' transformation procedures.

Clearly, we have

$$\boldsymbol{IF} = \boldsymbol{I} \tag{5.4}$$

and, from equation (5.2)

$$\boldsymbol{OF} = \boldsymbol{O} \tag{5.5}$$

$$\boldsymbol{UF} = \boldsymbol{U} \tag{5.6}$$

From equation (3.11) we see that a transformation is a monotonic mapping of $\hat{\mathscr{E}}_2$ into itself. We remember that a monotonic linear operator is bounded with respect to an order unit norm.

In conclusion, we have:

Proposition 6: The transformations are monotonic linear operators in the ordered linear space $\hat{\mathscr{E}}_2$ which map $\mathscr{E}_2$ into itself and have the property (5.6). They form a convex set $\boldsymbol{\mathscr{F}}$ in the space of the bounded linear operators in $\hat{\mathscr{E}}_2$. The operator product of two transformations is a transformation.

The concepts introduced above permit a clear discussion of the concept of state. A 'mathematical state' is a positive linear functional χ defined on the space $\hat{\mathscr{E}}_2$ with the property

$$\chi(\boldsymbol{U}) = 1 \tag{5.7}$$

A mathematical state χ will be called a 'physical state' if there is a transformation $\boldsymbol{F}$ of the form:

$$A\boldsymbol{F} = \chi(A)\boldsymbol{U}, \qquad A \in \hat{\mathscr{E}}_2 \tag{5.8}$$

If F is a transformation procedure corresponding to F from equation (3.9) we get

$$P(A, FS, 1) = P(AF, S, 1) \approx \chi(A), \qquad A \in \mathscr{E}_2 \tag{5.9}$$

and using equations (3.7) and (4.6) we obtain

$$P(B, FS, i) \approx \chi(B_i), \qquad B \in \mathscr{E}_n \tag{5.10}$$

We see that if we perform an arbitrary measurement procedure B in the ensemble FS, the probability of obtaining the result i does not depend on the initial ensemble of situations S. By means of the transformation procedure F we have been able to prepare, in a reproducible way, an ensemble of situations FS which provides a physical realisation of the mathematical state χ.

As we have explained in the Introduction, a procedure involves only a bounded region of the infinitely extended physical space. Consider the measurement procedure A performed after the transformation procedure F. If A lasts sufficiently long, it can be influenced by physical events which occur in a distant region and which cannot therefore be controlled by F. It follows that F cannot prepare a state in the sense explained above. In other words, the set $\mathscr{F}$ cannot contain elements of the form (5.8), namely transformations with one-dimensional range.

We have discussed the concept of physical state, even if we believe that physical states do not exist, because many approaches to quantum theory are based on this concept. In these treatments the statistical physical laws are expressed by a function $p(A, F)$ which gives the probability of obtaining the result 1 when we perform the measurement procedure $A \in \mathscr{E}_2$ on a state prepared by means of the transformation procedure F. With our notation, this means:

$$p(A, F) \approx P(AF, S, 1) \tag{5.11}$$

The order relation between measurement procedures is defined in terms of this function: the statement $A \leqslant B$ means that

$$p(A, F) \leqslant p(B, F) \tag{5.12}$$

for all the preparation procedures F.

This approach seems to avoid the difficulties we have found in Section 3 in defining the order relation with a sufficient degree of objectivity. In fact, the fundamental quantity $p(A, F)$ can be measured with an arbitrary accuracy by several experimenters in a reproducible way (if we disregard exceptional statistical fluctuations).

However, this advantage is illusory. Even if we forget our doubts about the existence of physical states and of preparation procedures, we have to remark that the function $p(A, F)$ does not contain all the empirical statements of the theory. Also the preliminary statement that F is a preparation procedure has a clearly empirical character and requires experimental verification.

Using our language, one has to test that the quantity (5.11) does not depend on S. It is clear that this test implies the same difficulties that we have found in Section 3 in our direct definition of the order relation.

In conclusion, we remark that both the physical states and the inhomogeneous Lorentz transformations appear in our approach as different idealisations of the concept of transformation. We remember that concepts previously distinct are expected to become interlaced and confused when we extend our field of experience (Bridgman, 1927).

6. *Compatibility of Measurements*

We think that the usual concept of compatibility between measurements is rather accurately described by the following definitions.

Definitions: A set $\mathscr{G}$ of measurements is called 'compatible' if, given any finite subset $\{A^{(1)}, \ldots, A^{(n)}\}$ of $\mathscr{G}$, there is a measurement C such that all the measurements $A^{(1)}, \ldots, A^{(n)}$ are statistical alterations of C, namely they can be written in the form:

$$A^{(r)} = \varphi^{(r)}C, \qquad r = 1, \ldots, n \tag{6.1}$$

If the measurement C can always be found in the set $\mathscr{G}$, we say that $\mathscr{G}$ is 'internally compatible'. A compatible set is called 'maximal' if it is not strictly contained in another compatible set. A set of measurement procedures is called compatible if the set composed of the corresponding measurements is compatible.

Two warnings are perhaps useful:

(a) Compatibility is not a binary relation: a set of pairwise compatible measurements is not necessarily compatible.
(b) The notion of compatibility should not be confused with the notion of joint feasibility. A set of procedures is called 'jointly feasible' if all of them can be performed in the same situation, or, more precisely, if putting together the corresponding prescriptions we obtain a set of prescriptions free of contradiction and ambiguity. A compatible set of measurement procedures is not necessarily jointly feasible.

Using Zorn's lemma, we see that every compatible set is contained in a maximal compatible set. It is clear that if $\mathscr{G}$ is a compatible set, adding to $\mathscr{G}$ a weighted mean of elements of $\mathscr{G}$ or a statistical alteration of an element of $\mathscr{G}$, we obtain another compatible set.

We say that a set $\mathscr{G}$ of measurements is a 'substructure' of the structure of measurements if it is closed with respect to the operations of weighted mean and of statistical alteration. In conclusion, we have the following result.

Proposition 7: A maximal compatible set is a substructure. Every compatible set is contained in a maximal compatible substructure.

The following proposition shows that we can always assume that the matrices $\varphi^{(r)}$ which appear in equation (6.1) have a simple form.

Proposition 8: The two measurements $\boldsymbol{A} \in \mathscr{E}_m$ and $\boldsymbol{B} \in \mathscr{E}_n$ are compatible if and only if there is an element $\boldsymbol{C} \in \mathscr{E}_{mn}$ such that:

$$\begin{cases} A_i = \sum_{k=0}^{n-1} C_{in+k} \\ B_k = \sum_{i=0}^{m-1} C_{in+k} \end{cases} \tag{6.2}$$

Proof: Due to our definition of compatibility, there is an element $\boldsymbol{D} \in \mathscr{E}_p$ with the property

$$\begin{cases} \boldsymbol{A} = \varphi \boldsymbol{D} \\ \boldsymbol{B} = \varphi' \boldsymbol{D} \end{cases} \tag{6.3}$$

Then equation (6.2) holds if we put

$$C_{in+k} = \sum_{r=0}^{p-1} \varphi_{ir}\varphi'_{kr} D_r \tag{6.4}$$

namely

$$\boldsymbol{C} = \varphi'' \boldsymbol{D} \tag{6.5}$$

where the probability matrix $\varphi'' \in \mathscr{M}_{mn,p}$ is defined by

$$\varphi''_{in+k,\, r} = \varphi_{ir}\varphi'_{kr} \tag{6.6}$$

If the whole structure of measurements is compatible, it is also internally compatible. The characterisation of the compatible structures of measurements is a complicated problem which we shall not discuss in detail. It is possible to find simple examples of compatible structures which are neither conical nor full. The following proposition gives a characterisation of the compatible conical full structures of measurements.

Proposition 9: The conical full structure of measurements generated by the ordered linear space $\hat{\mathscr{E}}_2$ is compatible if and only if the following condition is satisfied: Given two sets $\{\boldsymbol{A}_0, \ldots, \boldsymbol{A}_{m-1}\}$ and $\{\boldsymbol{B}_0, \ldots, \boldsymbol{B}_{n-1}\}$ of positive vectors with the property

$$\sum_r \boldsymbol{A}_r = \sum_r \boldsymbol{B}_r = \boldsymbol{U} \tag{6.7}$$

one can find a set of positive vectors $\{\boldsymbol{C}_0, \ldots, \boldsymbol{C}_{mn-1}\}$ in such a way that equations (6.2) are satisfied.

Proof: According to Proposition 8, the condition given above is equivalent to the requirement that all the pairs of measurements are compatible. Then one can show, by induction, that an arbitrary finite set of measurements is compatible.

This proposition permits us to give some interesting sufficient (but not necessary), conditions for the compatibility of a full conical structure of measurements.

(a) The space $\hat{\mathscr{E}}_2$ is an ordered algebra with unit U. This means that $\hat{\mathscr{E}}_2$ is both an ordered linear space and a real algebra and that the product of two positive elements is positive. In this case we have just to put

$$C_{in+k} = A_i B_k \tag{6.8}$$

As shown in Jameson (1970), this algebra is necessarily commutative and it is isomorphic to a dense subalgebra of the algebra of the continuous functions defined on a compact topological space Γ.

(b) The ordered linear space $\hat{\mathscr{E}}_2$ is a Riesz space, namely, whenever we have

$$A, B \leqslant C, D \tag{6.9}$$

one can find an element E with the property

$$A, B \leqslant E \leqslant C, D \tag{6.10}$$

In this case, the condition of Proposition 9 is satisfied, as shown in Jameson (1970).

(c) $\hat{\mathscr{E}}_2$ is a linear lattice, namely every pair of elements has a least upper bound and a greatest lower bound. This is an interesting special case of condition (b), as a linear lattice is also a Riesz space. Then, due to a theorem by Kakutani (Jameson, 1970), $\hat{\mathscr{E}}_2$ is isomorphic to a dense linear sublattice of the linear lattice of all the continuous functions defined on a compact topological space Γ. The order unit U corresponds to the constant function equal to one.

If the space $\hat{\mathscr{E}}_2$ is complete, conditions (a) and (c) are equivalent and $\hat{\mathscr{E}}_2$ is isomorphic to the space of all the continuous functions defined in Γ. A structure of measurements of this kind will be called 'classical' and the space Γ will be interpreted as a compactification of the 'phase space', as explained in detail in Giles (1970).

7. *Extended Structures of Measurements*

We have seen in Section 4 that the mathematical properties of the structure of measurements can be simplified by introducing some 'ideal' measurements which do not correspond to any measurement procedure. In this way we obtain an 'extension' of the original structure of measurements. We have shown how one can obtain full, conical and complete extensions. An extended structure of measurements can contain important information on physical theory. A smaller extension contains more information than a larger one.

In practice, it is not possible to give an explicit, detailed, list of all the measurement procedures. Moreover, this list would change every day due to the development of the experimental techniques. As a consequence, it is not possible to work with theories completely free of ideal measurements. In other words, in theoretical physics, one has always to deal with extended structures of measurements.

The ideal measurements have a very important role in the development of physics, as they provide a motivation and a guide in developing new, more refined, measurement procedures. For instance, the assumption that a certain quantity can be measured with an arbitrary accuracy implies the introduction of a class of ideal measurements. Some of them may become physical measurements with the improvement of measurement techniques.

It is important to remark that not all the extensions of the structure of measurements are physically meaningful. In fact, one has to take into account the existence of transformations: it must be possible to define in a consistent way the action of all the transformations on the ideal measurements. If this condition is satisfied, we say that the extension is 'acceptable'.

Proposition 10: The full conical structure generated by the ordered linear space $\hat{\mathscr{E}}_2$ is an acceptable extension of the original structure of measurements.

Proof: We have seen that a transformation is a monotonic linear operator in $\hat{\mathscr{E}}_2$ with the property (5.6) (see Proposition 6). If the structure of measurements is full and conical, an operator of this kind defines, through equation (5.3), for each value of n, a mapping of $\mathscr{E}_n$ into itself which has all the required formal properties.

Proposition 11: We consider the completion of the space $\hat{\mathscr{E}}_2$ with respect to the order unit norm. The closure of the cone $\mathscr{E}^+$ in the completion of $\hat{\mathscr{E}}_2$ defines in this space an antisymmetric and archimedean ordering. The ordered linear space defined in this way generates a complete conical full structure of measurements which is an acceptable extension of the original structure.

Proof: As the transformations are continuous linear operators, they can be extended to the completion of $\hat{\mathscr{E}}_2$. It is easy to show that these extended operators have all the required properties.

These propositions show that, independently from the properties of the set $\mathscr{F}$ of the transformations, one is always allowed to consider a complete full and conical extended structure of measurements. Practically all the current theories have structures of measurements of this kind.

If the set $\mathscr{F}$ has suitable properties, one can introduce further extensions of the structure of measurements. In this way one can obtain theories with simpler formal properties, but this advantage is paid for by a loss of physical information.

We remark that the introduction of ideal measurements affect the definition of compatibility. A set of measurements which is not compatible in the original structure may become compatible in an extended structure.

The considerations given in this section permit us to give a new formulation of the problem 'hidden variables' (von Neumann, 1932; Jauch, 1968; Capasso, Fortunato & Selleri, 1970). We say that a theory admits hidden variables if its structure of measurements has a classical acceptable extension. One can easily show that every structure of measurements has a classical extension, but it is not necessarily acceptable. This problem will be considered elsewhere.

Also, the problem of justifying the Hilbert space formalism (von Neumann, 1932) or the algebraic formalism (Giles, 1970; Segal, 1947; Haag & Kastler, 1964) of quantum mechanics can be formulated in a new way if one realises that these formalisms describe an extended structure of measurements.

Following Giles (1970), we define a class of structures of measurements which contain both the classical structures and those suggested by usual quantum theory.

Definition: A full conical structure of measurements is called a 'C^* structure' if the space $\hat{\mathscr{E}}_2$ is isomorphic to the space of the hermitean elements of a C^* algebra, with the positive elements defined in the usual way.

Then the special assumption which leads to the usual quantum theory can be formulated as follows: the structure of measurements has an acceptable C^* extension. As a C^* extension always exists, the essential point is that the extension has to be acceptable and we see that the set of transformations play an essential role.

Also the set $\mathscr{F}$ of transformations can be extended by introducing 'ideal' transformations. Of course, the ideal transformations must also map the sets $\mathscr{E}_n$ into themselves. If the structure of measurements is full and conical, we have only to require that the extended set $\mathscr{F}$ is a set of positive operators with the property (5.6), convex and closed with respect to the operator product. For instance, one is allowed to take the closure of the set $\mathscr{F}$ with respect to the uniform (norm) operator topology or to the strong operator topology.

It is not unreasonable to assume that the strong closure of $\mathscr{F}$ contains transformations with one-dimensional range of the type (5.8). These ideal transformations identify a mathematical state.

In conclusion, we think that these considerations show that the most basic features of a physical theory can be described in terms of the ordered linear space $\hat{\mathscr{E}}_2$ with antisymmetric archimedean ordering and with the order unit U and in terms of the convex set $\mathscr{F}$ of monotonic linear operators in $\hat{\mathscr{E}}_2$ which transform U into itself.

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Appendix

Proposition: $\hat{\mathscr{G}}$ and $\hat{\mathscr{F}}$ are affine spaces, $\hat{\mathscr{F}}\mathscr{G}$ is a convex subset of $\hat{\mathscr{G}}$ which generates it and α is a mapping of $\mathscr{G}$ into $\mathscr{F}$ which has the property

$$\alpha\Big(\sum_r a_r A^{(r)}\Big) = \sum_r a_r \alpha(A^{(r)}) \tag{A.1}$$

where the coefficients a_r satisfy equations (2.2) and (2.3). Then the mapping α can be extended univocally to an affine mapping of $\hat{\mathscr{G}}$ into $\hat{\mathscr{F}}$.

Proof: Every element of $\hat{\mathscr{G}}$ can be written in the form:

$$C = (1 + a)A - aB, \qquad a \geqslant 0, \qquad A, B, \in \mathscr{G} \tag{A.2}$$

We then have to put:

$$\alpha(C) = (1 + a)\alpha(A) - a\alpha(B) \tag{A.3}$$

If we also have

$$C = (1 + a')A' - a'B', \qquad a' \geqslant 0, \qquad A', B' \in \mathscr{G} \tag{A.4}$$

we can write

$$\frac{1 + a}{1 + a + a'} A + \frac{a'}{1 + a + a'} B' = \frac{1 + a'}{1 + a + a'} A' + \frac{a}{1 + a + a'} B \tag{A.5}$$

and from equation (A.1) we obtain

$$(1 + a)\alpha(A) - a\alpha(B) = (1 + a')\alpha(A') - a'\alpha(B') \tag{A.6}$$

We see that our definition (A.3) does not depend on the choice of the representation (A.2). It is easy to show that the extension of α obtained in this way is an affine mapping.

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